4.8. CONVEX PROGRAMMING

DEFINITIONS AND MATHEMATICAL RESULTS:

Set $C \subset \mathbb{R}^n$ is a *convex set*, if for all $\mathbf{x}$, $\mathbf{y} \in C$ and $0 \le \alpha \le 1$

$$\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in C.$$

Function $f: \mathbb{R}^n \to \mathbb{R}$ is a *convex function* in a set $C \subset \mathbb{R}^n$ if
for all $\mathbf{x}$, $\mathbf{y} \in C$ and $0 \le \alpha \le 1$

$$f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \le \alpha f(\mathbf{x}) + (1-\alpha)f(\mathbf{y})$$

Function $f: \mathbb{R}^n \to \mathbb{R}$ is a *concave function* in a set $C \subset \mathbb{R}^n$ if
for all $\mathbf{x}$, $\mathbf{y} \in C$ and $0 \le \alpha \le 1$

$$f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \ge \alpha f(\mathbf{x}) + (1-\alpha)f(\mathbf{y})$$

If f is twice continuously differntiable:
f is convex in C $\Leftrightarrow$ the Hessian matrix $H(\mathbf{x})$ of f is positive semidefinite for all $\mathbf{x} \in C$.

The set $C = \{\mathbf{x} \in \mathbb{R}^n \mid c_i(\mathbf{x}) \ge 0,\ i=1,...,m\}$ is convex, if the functions c_i are concave.

Or, the inequality reversed:
The set $C = \{\mathbf{x} \in \mathbb{R}^n \mid c_i(\mathbf{x}) \le 0,\ i=1,...,m\}$ is convex, if the functions c_i are convex

CONVEX PROGRAMMING PROBLEM

The problem

$$\text{(COP)} \qquad \begin{array}{lll} \text{minimize} & f(\mathbf{x}) = f(x_1,...,x_n) & \\ \text{subject to} & c_i(\mathbf{x}) = 0, & i = 1,...,m_e \\ & c_i(\mathbf{x}) \ge 0, & i = m_e+1,...,m \end{array}$$

is a *convex programming problem*, if $f(\mathbf{x})$ is convex, the equality constraint functions are linear and the inequality constraint functions are concave.

<u>Theorem 4.5.</u>
a) If $\mathbf{x^*}$ is a local minimum point of a convex programming problem, then $\mathbf{x^*}$ is a global minimum point.
b) If $\mathbf{x^*}$, $\mathbf{u^*}$ satisfy the necessary Kuhn-Tucker conditions of Theorem 4.1. for a convex programming problem, then $\mathbf{x^*}$ is a global minimum point of the problem.

Remarks on convex programming problems:
- No need for strategies for indefinite Hessian or unbounded subproblems in the augmented Lagrangian or SQP methods.
- No need for augmenting the Lagrangian to make $\mathbf{x^*}$ a minimum point.
- There is no practical method for determining if a general function is convex.

THE CUTTING PLANE METHOD FOR CONVEX PROGRAMMING PROBLEMS

In the *cutting plane method* the problem is transformed into a linear programming problem by linearizing the objective and the constraint functions about the current point. The method is also called a *sequential linear progamming method* (SLP-method). When the iterate $\mathbf{x_k}$ does not satisfy the constraints, new linearization is made about the point $\mathbf{x_k}$.

Consider an inequality contrained convex programming problem (f convex, c_i:s concave):

$$\begin{array}{ll} \text{minimize} & f(\mathbf{x}) = f(x_1,...,x_n) \\ \text{subject to} & c_i(\mathbf{x}) \geq 0, \quad i = 1,...,m \end{array}$$

Linearized functions:
$f(\mathbf{x}) \approx f(\mathbf{x_k}) + (\mathbf{x}-\mathbf{x_k})^T\nabla f(\mathbf{x_k})$
$c_i(\mathbf{x}) \approx c_i(\mathbf{x_k}) + (\mathbf{x}-\mathbf{x_k})^T\nabla c_i(\mathbf{x_k})$

The cutting plane algorithm

1. Choose a starting point $\mathbf{x_0}$. Set a tolerance for constraint satisfaction $\varepsilon>0$, maximum number of iterations kmax.. Set k=0.

2. Solve the problem LP_k:

$$\begin{array}{l} \min\ f(\mathbf{x_k}) + (\mathbf{x}-\mathbf{x_k})^T\nabla f(\mathbf{x_k}) \\ \text{subject to } c_i(\mathbf{x_k}) + (\mathbf{x}-\mathbf{x_k})^T\nabla c_i(\mathbf{x_k}) \geq \mathbf{0} \qquad i=1,...,m \end{array}$$

Denote the solution by $\mathbf{x_{k+1}}$.

3. Terminate if $c_i(\mathbf{x_{k+1}}) \geq -\varepsilon$ for all i=1,...,m or if k > kmax.

4. Find the most violated constraint j:

$$c_j(\boldsymbol{x_{k+1}}) = \min_i c_i(\boldsymbol{x_{k+1}})$$

and add the linearized constraint

$$c_j(\mathbf{x_{k+1}}) + (\mathbf{x}-\mathbf{x_{k+1}})^T\nabla c_j(\mathbf{x_{k+1}}) \geq \mathbf{0}$$

to the previous LP-problem.

5. Set m=m+1, k=k+1 and go to 2.

Remarks:
• All the iterates $\mathbf{x_k}$ are infeasible.
• The feasible set of the original problem is contained in the feasible polygon of the LP-problem.
• The method is quite efficient for convex problems with nearly linear functions.
• The method can be modified to solve *integer programming problems*.
• The new problem differs from the previous one by only one added constraint. If simplex method is used to solve the LP-subproblem, new constraints can be handled efficiently with a dual simplex method.
• The linearized problem may be unbounded. This can be prevented by adding bounds for all the variables $L_i \leq x_i \leq U_i$, i=1,...,n, or by defining the initial problem as

$$\min \; f(\mathbf{x_k}) + (\mathbf{x}-\mathbf{x_k})^T \nabla f(\mathbf{x_k})$$
$$\text{subject to } L_i \le x_i \le U_i,\; i=1,\ldots,n$$

Disregarding constant terms, the objective function can be written

$$\min \; \mathbf{x}^T \nabla f(\mathbf{x_k})$$

Example 4.6. Minimize $f(\mathbf{x}) = - x_1 - x_2$

subject to
$$2x_1 - x_2^2 - 1 \ge 0$$
$$9 - 0.8x_1^2 - 2x_2 \ge 0$$
$$x_1 \ge 0$$
$$x_2 \ge 0$$

Start with a box-constrained problem:

$$\min f(\mathbf{x})$$
s.t.
$$0 \le x_1 \le 5$$
$$0 \le x_2 \le 4$$

The constraint functions are concave and the feasible region is contained in the initial rectangle.