4.3. QUADRATIC PROGRAMMING

Problem QP: minimize $f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{G}\boldsymbol{x} + \boldsymbol{c}^T\boldsymbol{x}$

subject to $\mathbf{Ax} \geq \mathbf{b}$

a) Most of the algorithms for QP are **active set projection methods**. Advantage can be taken of the special structure of the problem, e.g. that the Hessian is constant. Special strategies are needed for an indefinite Hessian G.

The search direction is obtained from the projected Newton equations

$$\mathbf{Z_k^T G Z_k q = -Z_k^T \nabla f(x_k)}$$

by setting $\mathbf{p_k} = \mathbf{Z_k q}$, where $\mathbf{Z_k}$ is the matrix whose columns form a basis for the set of vectors orthogonal to the active rows of **A**.

b) Other methods use the special structure of the Lagrangian to transform the problem to a linear programming problem. Let us make an additional restriction that $\mathbf{x} \geq \mathbf{0}$ (typical in most applications of quadratic optimization).

$$L(\boldsymbol{x},\boldsymbol{u}) = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{G}\boldsymbol{x} + \boldsymbol{c}^T\boldsymbol{x} - \boldsymbol{u}^T(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}) - \boldsymbol{v}^T\boldsymbol{x}$$

$$\nabla_x L(\mathbf{x},\mathbf{u}) = \mathbf{Gx} + \mathbf{c} - \mathbf{A^T u} - \mathbf{v}$$

The Kuhn-Tucker conditions are

$$\mathbf{Ax} \geq \mathbf{b}$$
$$\mathbf{Gx + c - A^T u - v = 0}$$
$$\mathbf{u^T(Ax-b) = 0}$$
$$\mathbf{v^t x = 0}$$
$$\mathbf{u} \geq \mathbf{0}$$
$$\mathbf{v} \geq \mathbf{0}$$
$$\mathbf{x} \geq \mathbf{0}$$

Using slack variables $\mathbf{y} = \mathbf{Ax} - \mathbf{b}$, we need to solve

$$\mathbf{Ax - y = b}$$
$$\mathbf{Gx - A^T u - v = -c}$$
$$\mathbf{u^T y = 0}$$
$$\mathbf{v^t x = 0}$$
$$\mathbf{u} \geq \mathbf{0},\ \mathbf{y} \geq \mathbf{0},\ \mathbf{x} \geq \mathbf{0},\ \mathbf{y} \geq \mathbf{0}$$

All the equations are linear and this system can be solved with the first phase of a **two phase simplex method** (because it has no objective function): artificial variables are added to each equation and their sum is minimized to get a feasible basic solution.

Wolfe's method and **Beale's method** use LP-based strategies like the one above.

These methods are not suitable for large scale problems, the extra variables increasing the dimensionality of the problem.